

FIELD THEORY CORRELATORS AND STRING THEORY

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Abstract It appears that string-M-theory is the only viable candidate for a complete theory of matter. It must therefore contain both gravity and QCD. What is particularly surprising is the recent conjecture that strongly coupled QCD matrix elements can be evaluated through a duality with weakly coupled gravity. To date there has been no direct verification of this conjecture by Maldacena because of the difficulty of direct strong coupling calculations in gauge theories. We report here on some progress in evaluating a gauge-invariant correlator in the non-perturbative regime in two and three dimensions in SYM theories. The calculations are made using supersymmetric discrete light-cone quantization (SDLCQ). We consider a Maldacena-type conjecture applied to the near horizon geometry of a D1-brane in the supergravity approximation, solve the corresponding $\mathcal{N} = (8, 8)$ SYM theory in two dimensions, and evaluate the correlator of the stress-energy tensor. Our numerical results support the Maldacena conjecture and are within 10-15% of the predicted results. We also present a calculation of the stress-energy correlator in $\mathcal{N} = 1$ SYM theory in 2+1 dimensions. While there is no known duality relating this theory to supergravity, the theory does have massless BPS states, and the correlator gives important information about the BPS wave function in the non-perturbative regime.

Introduction

Recently the conjecture that certain field theories admit concrete realizations as string theories on particular backgrounds has caused a lot of excitement. The original Maldacena conjecture [1] asserts that the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in 3+1 dimensions is equivalent to Type IIB string theory on an $AdS_5 \times S^5$ background. However, more recently, other string/field theory correspondences have been conjectured. Attempts to rigorously test these conjectures have met with only limited success, because our understanding of both sides of the correspondences is usually insufficient. The main obstacle is that at the point of correspondence we want the curvature of space-time to be small in order to use the supergravity approximation to string theory. This requires a *non-perturbative* calculation on the field theory side. We use the method, SDLCQ, in the corresponding non-perturbative regime. SDLCQ, or Supersymmetric Discretized Light-Cone Quantization, is a non-perturbative method for solving bound-state problems that has been shown to have excellent convergence properties[2].

Aside from our numerical solutions, there has been very little work on solving SYM theories using methods that might be described as being from first principles. While selected properties of these theories have been investigated, one needs the complete solution of the theory to calculate the correlators. By a “complete solution” we mean the spectrum and the wave functions of the theory in some well-defined basis. The SYM theories that are needed for the correspondence with supergravity and string theory have typically a high degree of supersymmetry and therefore a large number of fields. The number of fields significantly increases the size of the numerical problem. Therefore, when presenting the first calculation of correlators in 2+1 dimensions, we consider only $\mathcal{N} = 1$ SYM.

An important step in these considerations is to find an observable that can be computed relatively easily on both sides of a string/field theory correspondence. It turns out that the correlation function of a gauge invariant operator is a well-behaved object in this sense. We chose the stress-energy tensor $T^{\mu\nu}$ as this operator and will construct this observable in the supergravity approximation to string theory and perform a non-perturbative SDLCQ calculation of this correlator on the field theory side.

1. String Theory

String theory contains solitons, the so-called D-branes on which modes can propagate as well as in the bulk. While in general these modes

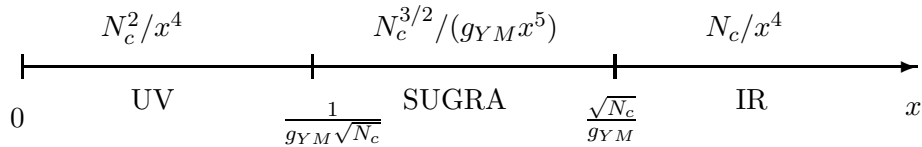


Figure 1. Phase diagram of two-dimensional $\mathcal{N} = (8, 8)$ SYM: the theory flows from a CFT in the UV to a conformal σ -model in the IR. The SUGRA approximation is valid in the intermediate range of distances, $1/g_{YM}\sqrt{N_c} < x < \sqrt{N_c}/g_{YM}$.

couple, there exists a limit in which the bulk modes decouple from the modes on the D-brane; this is typically a low energy limit. In this limit the theory on N D p -branes, separated by at most sub-stringy distances becomes a supersymmetric $SU(N)$ Yang-Mills theory. As the D-branes carry mass and charge, they can excite gravity modes in the bulk, for which in supergravity there exist equivalent solutions. One thus has a string/field theory correspondence. Naively, one would think that supergravity can only describe the large distance behavior of fields, but it turns out that one can trust these solutions as long as the curvature is small compared to the string scale. In this sense, the large N limit is a valid description[3].

The most prominent string/field theory correspondence is the so-called Maldacena conjecture[1], which assures that the conformal $\mathcal{N} = 4$ SYM in 3+1 dimensions, is equivalent to a type IIB string theory on a $AdS_5 \times S^5$ background. In the more general case of non-conformal theories, it turns out that a black p -brane solution, stretching in $p + 1$ spacetime dimensions, of supergravity will correspond to a supersymmetric Yang-Mills theory in $p + 1$ dimensions [3].

One can test these string/field theory correspondences, if one is able to construct and evaluate observables on both the string and the field theory regimes. Although this at first seems a hard task, because typically the small curvature regime of string theory, where the supergravity approximation allows quantitative calculations, falls into the strong coupling regime of the field theory side. We shall see that we can come up with scenarios where we can evaluate the field theory observable non-perturbatively.

1.1. Two-dimensional correlation functions from supergravity

It is instructive to take a closer look on the expected properties of $\mathcal{N} = (8, 8)$ SYM in two dimensions, before we proceed to technical details on the string theory side. In the extreme ultra-violet (UV) this theory

is conformally free and has a central charge $c_{UV} = N_c^2$. Perturbation theory in turn will be valid for small effective couplings $g = g_{YM}\sqrt{N_c}x$, where x is a space coordinate. For large distances, in the far infrared (IR), the theory becomes a conformal σ -model with target space $(R^8)^{N_c}/S_{N_c}$. The central charge is $c_{IR} = N_c$. It is a bit more involved to show that here perturbation theory breaks down when $x \sim \sqrt{N_c}/g_{YM}$, see *e.g.* Ref. [3].

The intermediate region, $1/g_{YM}\sqrt{N_c} < x < \sqrt{N_c}/g_{YM}$, where no perturbative field theoretical description is possible, is fortunately exactly the region which is accessible to string theory; or rather, to the supergravity (SUGRA) approximation to Type IIB string theory on a special background. It is that of the near horizon geometry of a D1-brane in the string frame, which has the metric

$$\begin{aligned} ds^2 &= \alpha' \hat{g}_{YM} \left(\frac{U^3}{g_s^2} dx_{\parallel}^2 + \frac{dU^2}{U^3} + U d\Omega_{8-p}^2 \right) \\ e^{\phi} &= \frac{2\pi g_{YM}^2}{U^3} \hat{g}_{YM}, \end{aligned} \quad (1)$$

where we defined $\hat{g}_{YM} \equiv 8\pi^{3/2} g_{YM} \sqrt{N_c}$. In the description of the computation of the two-point function we follow Ref. [4]. The correlator has been derived in Ref. [5], being itself a generalization of Refs. [6, 7].

First, we need to know the action of the diagonal fluctuations around this background to the quadratic order. We would like to use the analogue of Ref. [8] for our background, Eq. (1), which is not (yet) available in the literature. However, we can identify some diagonal fluctuating degrees of freedom by following the work on black hole absorption cross-sections [9, 10]. One can show that the fluctuations parameterized like

$$\begin{aligned} ds^2 &= \left(1 + f(x^0, U) + g(x^0, U)\right) g_{00} (dx^0)^2 \\ &\quad + \left(1 + 5f(x^0, U) + g(x^0, U)\right) g_{11} (dx^1)^2 \\ &\quad + \left(1 + f(x^0, U) + g(x^0, U)\right) g_{UU} dU^2 \\ &\quad + \left(1 + f(x^0, U) - \frac{5}{7}g(x^0, U)\right) g_{\Omega\Omega} d\Omega_7^2 \\ e^{\phi} &= \left(1 + 3f(x^0, U) - g(x^0, U)\right) e^{\phi_0}, \end{aligned} \quad (2)$$

satisfy the following equations of motion

$$\begin{aligned} f''(U) &= -\frac{7}{U} f'(U) + \frac{g_s^2 k^2}{U^6} f(U) \\ g''(U) &= -\frac{7}{U} g'(U) + \frac{72}{U^2} g(U) + \frac{g_s^2 k^2}{U^6} g(U). \end{aligned} \quad (3)$$

Without loss of generality we have assumed here that these fluctuations vary only along the x^0 direction of the world volume coordinates, and behave like a plane wave. One can interpret a D1-brane as a black hole in nine dimensions. The fields $f(U)$ and $g(U)$ are the minimal set of fixed scalars in this black hole geometry. In ten dimensions, however, we see that they are really part of the gravitational fluctuation. Consequently, we expect that they are associated with the stress-energy tensor in the operator field correspondence of Refs. [6, 7]. In the case of the correspondence between $\mathcal{N}=4$ SYM field theory and string theory on an $AdS_5 \times S^5$ background, the superconformal symmetry allows for the identification of operators and fields in short multiplets [11]. In the present case of a D1-brane, we do not have superconformal invariance, and this technique is not applicable. Actually, we expect all fields of the theory consistent with the symmetry of a given operator to mix. The large distance behavior should then be dominated by the contribution with the longest range. The field $f(k^0, U)$ appears to be the one with the longest range since it is the lightest field.

Eq. (3) for $f(U)$ can be solved explicitly

$$f(U) = U^{-3} K_{3/2} \left(\frac{\hat{g}_{YM}}{2U^2} k \right), \quad (4)$$

where $K_{3/2}(x)$ is a modified Bessel function. If we take $f(U)$ to be the analogue of the minimally coupled scalar, we can construct the flux factor

$$\begin{aligned} \mathcal{F} &= \lim_{U_0 \rightarrow \infty} \frac{1}{2\kappa_{10}^2} \sqrt{g} g^{UU} e^{-2(\phi - \phi_\infty)} \partial_U \log(f(U))|_{U=U_0} \\ &= \frac{NU_0^2 k^2}{2g_{YM}^2} - \frac{N^{3/2} k^3}{4g_{YM}} + \dots \end{aligned} \quad (5)$$

up to a numerical coefficient of order one which we have suppressed. We see that the leading non-analytic contribution in k^2 is due to the k^3 term. Fourier transforming the latter yields

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \frac{N^{\frac{3}{2}}}{g_{YM} x^5}. \quad (6)$$

This is in line with the discussion at the beginning of this section. We expect to deviate from the trivial $(1/x^4)$ scaling behavior of the correlator at $x_1 = 1/g_{YM}\sqrt{N_c}$ and $x_2 = \sqrt{N_c}/g_{YM}$. This yields the phase diagram in Fig. 1. It is interesting to note that the entire N_c hierarchy is consistent in the sense of Zamolodchikov's c-theorem, which assures that the central charges obey $c(x) > c(y)$, whenever $x < y$ [12].

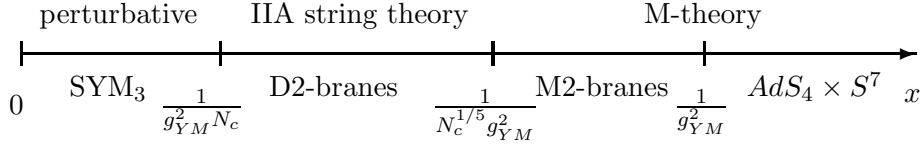


Figure 2. Phase diagram of three-dimensional $\mathcal{N} = (8, 8)$ SYM: the theory flows from a perturbative SYM in the UV to a M-theory on $AdS_4 \times S^7$. The SUGRA approximation is valid in the intermediate range of distances, $1/g_{YM}^2 N_c < x < 1/g_{YM}^2 N_c^{1/5}$.

1.2. D2-branes and three-dimensional SYM

As stated in the introduction, in an analogous way one can show that a system of $D2$ -branes corresponds in a certain limit to a Yang-Mills theory in three dimensions. It is again a $\mathcal{N} = (8, 8)$ supersymmetric theory. Unfortunately, an observable like the correlation function of the stress-energy tensor has not yet been calculated for this theory. However, there are encouraging results both on the string and on the field theory side of the correspondence [3, 13].

We describe the phase-diagram of three-dimensional SYM with 16 supercharges here, following Ref. [3]. Later we will present a non-perturbative field theory calculation within the SDLCQ framework. The latter calculation is, however, of a theory with an $\mathcal{N} = 1$ supersymmetry. This theory might nevertheless share some features with the full $\mathcal{N} = (8, 8)$ theory, *cf.* also the results of two-dimensional SYM with different supersymmetries in Sec. 2.

It can be argued that the theory has to be described by different degrees of freedom at different energy scales. At large N_c , one can use perturbation theory of SYM(2+1) in the far ultra-violet, *i.e.* at small distances. The supergravity solution, which in this regime is an approximation to type IIA string theory with D2-branes, can be trusted at intermediate distances r , $1/g_{YM}^2 N_c < r < 1/g_{YM}^2 N_c^{1/5}$. It has been conjectured that for large distances, $r > 1/g_{YM}^2 N_c^{1/5}$, an M-theory description is appropriate, while in the far infrared, $r \gg 1/g_{YM}^2$, this theory is equivalent to M-theory on an $AdS_4 \times S^7$ background, dual to a CFT with an $SO(8)$ R-symmetry. This picture is compiled in Fig. 2.

2. Field theory correlators and SDLCQ

Discretized Light-Cone Quantization (DLCQ) preserves supersymmetry at every stage of the calculation if the supercharge rather than the Hamiltonian is diagonalized [14, 15]. The framework of supersymmetric DLCQ (SDLCQ) allows one to use the advantages of light-cone quantiza-

tion (*e.g.* a simpler vacuum) together with the excellent renormalization properties guaranteed by supersymmetry.

The technique of (S)DLCQ was reviewed in Ref. [16], so we can be brief here. The basic idea of light-cone quantization is to parameterize space-time using light-cone coordinates

$$x^{\pm} \equiv \frac{1}{\sqrt{2}} (x^0 \pm x^1), \quad (7)$$

and to quantize the theory making x^+ play the role of time. In the discrete light-cone approach, we require the momentum $p_- = p^+$ along the x^- direction to take on discrete values in units of p^+/K where p^+ is the conserved total momentum of the system. The integer K is the so-called harmonic resolution, and plays the role of a discretization parameter. One can think of this discretization as a consequence of compactifying the x^- coordinate on a circle with a period $2L = 2\pi K/p^+$. The advantage of discretizing on the light cone is the fact that the dimension of the Hilbert space becomes finite. Therefore, the Hamiltonian is a finite-dimensional matrix, and its dynamics can be solved explicitly. In SDLCQ one makes the DLCQ approximation to the supercharges Q^i . Surprisingly, also the discrete representations of Q_i satisfy the supersymmetry algebra. Therefore SDLCQ enjoys the improved renormalization properties of supersymmetric theories. To recover the continuum result, K has to go to infinity. We find that SDLCQ usually converges much faster than the naive DLCQ.

In the three-dimensional case we also discretize the transverse momentum along the direction x^{\perp} ; however, it is treated in a fundamentally different way. The transverse resolution is T , and we think of the theory as being compactified on a transverse circle of length l . Therefore, the transverse momentum is cut off at $\pm 2\pi T/l$ and discretized in units of $2\pi/l$. Removal of this transverse momentum cutoff therefore corresponds to taking the transverse resolution T to infinity.

Let us now review these ideas in the context of a specific super-Yang-Mills (SYM) theory. Actually, it turns out that the two-dimensional SYM is essentially 'included' in the three-dimensional case [17], in the sense that in the weak coupling limit the spectrum of the three-dimensional theory is that of the lower dimensional theory. We therefore describe here only the 'more general' three-dimensional theory and hint at the differences and changes that are to make to recover the two-dimensional theory. We start with $2+1$ dimensional $\mathcal{N} = 1$ super-Yang-Mills theory [18] defined on a space-time with one transverse dimension compactified

on a circle. The action is

$$S = \int d^2x \int_0^l dx_\perp \text{tr} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\bar{\Psi} \gamma^\mu D_\mu \Psi \right). \quad (8)$$

After introducing the light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, decomposing the spinor Ψ in terms of chiral projections

$$\psi = \frac{1 + \gamma^5}{2^{1/4}} \Psi, \quad \chi = \frac{1 - \gamma^5}{2^{1/4}} \Psi \quad (9)$$

and choosing the light-cone gauge $A^+ = 0$, we obtain the action in the form

$$\begin{aligned} S = \int dx^+ dx^- \int_0^l dx_\perp \text{tr} \left[\frac{1}{2} (\partial_- A^-)^2 + D_+ \phi \partial_- \phi + i\psi D_+ \psi + \right. \\ \left. + i\chi \partial_- \chi + \frac{i}{\sqrt{2}} \psi D_\perp \phi + \frac{i}{\sqrt{2}} \phi D_\perp \psi \right]. \end{aligned} \quad (10)$$

A simplification of the light-cone gauge is that the non-dynamical fields A^- and χ may be explicitly solved from their Euler-Lagrange equations of motion

$$A^- = \frac{g_{\text{YM}}}{\partial_-^2} J = \frac{g_{\text{YM}}}{\partial_-^2} (i[\phi, \partial_- \phi] + 2\psi\psi), \quad \chi = -\frac{1}{\sqrt{2}\partial_-} D_\perp \psi. \quad (11)$$

These expressions may be used to express any operator in terms of the physical degrees of freedom only. In particular, the light-cone energy, P^- , and momentum operators, P^+, P^\perp , corresponding to translation invariance in each of the coordinates x^\pm and x_\perp may be calculated explicitly as

$$P^+ = \int dx^- \int_0^l dx_\perp \text{tr} \left[(\partial_- \phi)^2 + i\psi \partial_- \psi \right], \quad (12)$$

$$P^- = \int dx^- \int_0^l dx_\perp \text{tr} \left[-\frac{g_{\text{YM}}^2}{2} J \frac{1}{\partial_-^2} J - \frac{i}{2} D_\perp \psi \frac{1}{\partial_-} D_\perp \psi \right], \quad (13)$$

$$P_\perp = \int dx^- \int_0^l dx_\perp \text{tr} \left[\partial_- \phi \partial_\perp \phi + i\psi \partial_\perp \psi \right]. \quad (14)$$

The light-cone supercharge in this theory is a two-component Majorana spinor, and may be conveniently decomposed in terms of its chiral projections

$$Q^+ = 2^{1/4} \int dx^- \int_0^l dx_\perp \text{tr} \left[\phi \partial_- \psi - \psi \partial_- \phi \right], \quad (15)$$

$$Q^- = 2^{3/4} \int dx^- \int_0^l dx_\perp \text{tr} \left[2\partial_\perp \phi \psi + g_{\text{YM}} (i[\phi, \partial_- \phi] + 2\psi\psi) \frac{1}{\partial_-} \psi \right].$$

The action (9) gives the following canonical (anti-)commutation relations for propagating fields for large N_c at equal x^+ :

$$\begin{aligned} [\phi_{ij}(x^-, x_\perp), \partial_- \phi_{kl}(y^-, y_\perp)] &= \{\psi_{ij}(x^-, x_\perp), \psi_{kl}(y^-, y_\perp)\} \\ &= \frac{1}{2} \delta(x^- - y^-) \delta(x_\perp - y_\perp) \delta_{il} \delta_{jk}. \end{aligned} \quad (16)$$

Using these relations one can check the supersymmetry algebra

$$\{Q^\pm, Q^\pm\} = 2\sqrt{2}P^\pm, \quad \{Q^+, Q^-\} = -4P_\perp. \quad (17)$$

In solving for mass eigenstates, we will consider only states which have vanishing transverse momentum, which is possible since the total transverse momentum operator is kinematical. Strictly speaking, on a transverse cylinder, there are separate sectors with total transverse momenta $2\pi N_\perp/L$; we consider only one of them, $N_\perp = 0$. On such states, the light-cone supercharges Q^+ and Q^- anti-commute with each other, and the supersymmetry algebra is equivalent to the $\mathcal{N} = (1, 1)$ supersymmetry of the dimensionally reduced (*i.e.*, two-dimensional) theory [14]. Moreover, in the $P_\perp = 0$ sector, the mass squared operator M^2 is given by $M^2 = 2P^+P^-$.

As we mentioned earlier, in order to render the bound-state equations numerically tractable, the transverse momenta of partons must be truncated. First, we introduce the Fourier expansion for the fields ϕ and ψ , where the transverse space-time coordinate x_\perp is periodically identified

$$\phi_{ij}(0, x^-, x_\perp) = \frac{1}{\sqrt{2\pi l}} \sum_{n^\perp=-\infty}^{\infty} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \quad (18)$$

$$\times \left[a_{ij}(k^+, n^\perp) e^{-ik^+x^- - i\frac{2\pi n^\perp}{l}x_\perp} + a_{ji}^\dagger(k^+, n^\perp) e^{ik^+x^- + i\frac{2\pi n^\perp}{l}x_\perp} \right],$$

$$\psi_{ij}(0, x^-, x_\perp) = \frac{1}{2\sqrt{\pi l}} \sum_{n^\perp=-\infty}^{\infty} \int_0^\infty dk^+ \quad (19)$$

$$\times \left[b_{ij}(k^+, n^\perp) e^{-ik^+x^- - i\frac{2\pi n^\perp}{l}x_\perp} + b_{ji}^\dagger(k^+, n^\perp) e^{ik^+x^- + i\frac{2\pi n^\perp}{l}x_\perp} \right].$$

Substituting these into the (anti-)commutators (16), one finds

$$\begin{aligned} [a_{ij}(p^+, n_\perp), a_{lk}^\dagger(q^+, m_\perp)] &= \{b_{ij}(p^+, n_\perp), b_{lk}^\dagger(q^+, m_\perp)\} \\ &= \delta(p^+ - q^+) \delta_{n_\perp, m_\perp} \delta_{il} \delta_{jk}. \end{aligned}$$

The supercharges then take the following form:

$$Q^+ = i2^{1/4} \sum_{n^\perp \in \mathbf{Z}} \int_0^\infty dk \sqrt{k} \quad (20)$$

$$\begin{aligned}
& \times \left[b_{ij}^\dagger(k, n^\perp) a_{ij}(k, n^\perp) - a_{ij}^\dagger(k, n^\perp) b_{ij}(k, n^\perp) \right], \\
Q^- = & \frac{2^{7/4} \pi i}{l} \sum_{n^\perp \in \mathbf{Z}} \int_0^\infty dk \frac{n^\perp}{\sqrt{k}} \\
& \times \left[a_{ij}^\dagger(k, n^\perp) b_{ij}(k, n^\perp) - b_{ij}^\dagger(k, n^\perp) a_{ij}(k, n^\perp) \right] + \\
& + \frac{i 2^{-1/4} g_{\text{YM}}}{\sqrt{l} \pi} \sum_{n_i^\perp \in \mathbf{Z}} \int_0^\infty dk_1 dk_2 dk_3 \delta(k_1 + k_2 - k_3) \delta_{n_1^\perp + n_2^\perp, n_3^\perp} \left\{ \right. \\
& \frac{1}{2\sqrt{k_1 k_2}} \frac{k_2 - k_1}{k_3} [a_{ik}^\dagger(k_1, n_1^\perp) a_{kj}^\dagger(k_2, n_2^\perp) b_{ij}(k_3, n_3^\perp) \\
& \quad - b_{ij}^\dagger(k_3, n_3^\perp) a_{ik}(k_1, n_1^\perp) a_{kj}(k_2, n_2^\perp)] \\
& \frac{1}{2\sqrt{k_1 k_3}} \frac{k_1 + k_3}{k_2} [a_{ik}^\dagger(k_3, n_3^\perp) a_{kj}(k_1, n_1^\perp) b_{ij}(k_2, n_2^\perp) \\
& \quad - a_{ik}^\dagger(k_1, n_1^\perp) b_{kj}^\dagger(k_2, n_2^\perp) a_{ij}(k_3, n_3^\perp)] \\
& \frac{1}{2\sqrt{k_2 k_3}} \frac{k_2 + k_3}{k_1} [b_{ik}^\dagger(k_1, n_1^\perp) a_{kj}^\dagger(k_2, n_2^\perp) a_{ij}(k_3, n_3^\perp) \\
& \quad - a_{ij}^\dagger(k_3, n_3^\perp) b_{ik}(k_1, n_1^\perp) a_{kj}(k_2, n_2^\perp)] \\
& \left. \left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} \right) [b_{ik}^\dagger(k_1, n_1^\perp) b_{kj}^\dagger(k_2, n_2^\perp) b_{ij}(k_3, n_3^\perp) \right. \right. \\
& \quad \left. \left. + b_{ij}^\dagger(k_3, n_3^\perp) b_{ik}(k_1, n_1^\perp) b_{kj}(k_2, n_2^\perp) \right] \right\}.
\end{aligned} \tag{21}$$

We now perform the truncation procedure; namely, in all sums over the transverse momenta n_i^\perp appearing in the above expressions for the supercharges, we restrict summation to the following allowed momentum modes: $n_i^\perp = 0, \pm 1 \dots \pm T$. Note that this prescription is symmetric, in the sense that there are as many positive modes as there are negative ones. In this way we retain parity symmetry in the transverse direction. The longitudinal momenta $k_i = n_i \pi / L$ are restricted by the longitudinal resolution according to $K = \sum_i n_i$.

The two-dimensional supercharges are essentially recovered, when we put n_\perp to zero. In particular, the first term of the supercharge Q^- , Eq. (21), is absent in this case. Additionally, we have to adjust the normalization constants in front of the expressions for the supercharges.

2.1. Two dimensional correlators

Using SDLCQ, we can reproduce the SUGRA scaling relation, Eq. (6), fix the numerical coefficient, and calculate the cross-over behavior at $1/g_{\text{YM}} \sqrt{N_c} < x < \sqrt{N_c}/g_{\text{YM}}$. To exclude subtleties, *nota bene* issues of

zero modes, we checked our results against the free fermion and the 't Hooft model and found consistent results.

Let us now focus on the theory in two dimensions. We would like to compute a general expression for the correlator of the form $F(x^-, x^+) = \langle \mathcal{O}(x^-, x^+) \mathcal{O}(0, 0) \rangle$. In DLCQ one fixes the total momentum in the x^- direction, and it is natural to compute the Fourier transform and express it in a spectrally decomposed form

$$\begin{aligned} \tilde{F}(P_-, x^+) &= \frac{1}{2L} \langle \mathcal{O}(P_-, x^+) \mathcal{O}(-P_-, 0) \rangle \\ &= \sum_n \frac{1}{2L} \langle 0 | \mathcal{O}(P_-) | n \rangle e^{-iP_-^+ x^+} \langle n | \mathcal{O}(-P_-, 0) | 0 \rangle. \end{aligned} \quad (22)$$

The form of the correlation function in position space is then recovered by Fourier transforming with respect to $P_- = K\pi/L$. We can continue to Euclidean space by taking $r = \sqrt{2x^+x^-}$ to be real. The result for the correlator of the stress-energy tensor is

$$\begin{aligned} F(x^-, x^+) &= \sum_n \left| \frac{L}{\pi} \langle n | T^{++}(-K) | 0 \rangle \right|^2 \left(\frac{x^+}{x^-} \right)^2 \\ &\quad \times \frac{M_n^4}{8\pi^2 K^3} K_4 \left(M_n \sqrt{2x^+x^-} \right), \end{aligned} \quad (23)$$

where M_i is a mass eigenvalue and $K_4(x)$ is the modified Bessel function of order 4. Note that this quantity depends on the harmonic resolution K , but involves no other unphysical quantities. In particular, the expression is independent of the box length L .

The matrix element $(L/\pi) \langle 0 | T^{++}(K) | i \rangle$ can be substituted directly to give an explicit expression for the two-point function. We see immediately that the correlator has the correct small- r behavior, for in that limit, it asymptotes to

$$\left(\frac{x^-}{x^+} \right)^2 F(x^-, x^+) = \frac{N_c^2(2n_b + n_f)}{4\pi^2 r^4} \left(1 - \frac{1}{K} \right),$$

which we expect for the theory of $n_b(n_f)$ free bosons (fermions) at large K . On the other hand, the contribution to the correlator from strictly massless states is given by

$$\left(\frac{x^-}{x^+} \right)^2 F(x^-, x^+) = \frac{6}{K^3 \pi^2 r^4} \sum_i \left| \frac{L}{\pi} \langle 0 | T^{++}(K) | i \rangle \right|_{M_i=0}^2.$$

It is important to notice that this $1/r^4$ behavior at large r is *not* the one we are looking for at large r . First of all, we do not expect any

massless physical bound state in this theory, and, additionally, it has the wrong N_c dependence. Relative to the $1/r^4$ behavior at small r , the $1/r^4$ behavior at large r that we expect is down by a factor of $1/N_c$. This behavior is suppressed because we are performing a large- N_c calculation. All we can hope is to see the transition from the $1/r^4$ behavior at small r to the region where the correlator behaves like $1/r^5$.

2.1.1 The $\mathcal{N} = (1, 1)$ theory. Although it is the $\mathcal{N} = (8, 8)$ theory in which we are ultimately interested in, we can, nevertheless, perform the computation of the correlation function in models with less supersymmetry. The evaluation of the correlator for the stress energy tensor in the $\mathcal{N} = (8, 8)$ theory is especially hard because of the many degrees of freedom due to the large number of supercharges in that theory. We will show in Sec. 2.1.3 how to overcome this obstacle by exploiting a residual ‘flavor’ symmetry of the theory. To see how our numerical method works without these complications, it might be worthwhile to study the theory with supercharges $(1, 1)$. In the next section we will briefly cover the theory with a $\mathcal{N} = (2, 2)$ supersymmetry.

It has been argued that the $\mathcal{N} = (1, 1)$ SYM theory does not exhibit dynamical supersymmetry breaking. A physicist’s proof that supersymmetry is not spontaneously broken in this theory was given in Ref. [4]. This theory is also believed not to be confining [19][20], and is therefore expected to exhibit non-trivial infra-red dynamics. The SDLCQ of the 1+1 dimensional model with $\mathcal{N} = (1, 1)$ supersymmetry was solved in Refs. [14, 21], and we apply these results directly in order to compute (23). For simplicity, we work at large N_c . The spectrum of this theory at finite K consists of $2K - 2$ exactly massless states, *i.e.* $K - 1$ massless bosons, and their superpartners, accompanied by large numbers of massive states separated by a gap. There is numerical evidence that this gap closes in the continuum limit. At finite N_c , we expect the degeneracy of $2K - 2$ exactly massless states to be broken, giving rise to precisely a continuum of states starting at $M = 0$ as expected.

The stress-energy correlator of this theory for various values of the harmonic resolution K , is shown in Fig. 3(a). We find the curious feature that it asymptotes to the inverse power law c/r^4 for large r . This behavior comes about due to the coupling $\langle 0|T^{++}|n\rangle$ with exactly massless states $|n\rangle$. The contribution to (23) from strictly massless states are given by

$$\left(\frac{x^-}{x^+}\right)^2 F(x^-, x^+) = \left| \frac{L}{\pi} \langle 0|T^{++}(k)|n\rangle \right|^2 \frac{M_n^4}{8\pi^2 k^3} K_4(M_n r) \Big|_{M_n=0} \quad (24)$$

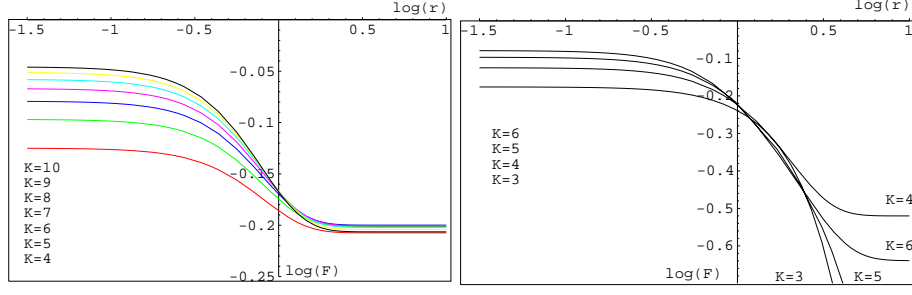


Figure 3. Log-Log plots of the two-dimensional correlation function $\langle T^{++}(x)T^{++}(0) \rangle \left(\frac{x^-}{x^+}\right)^2 \frac{4\pi^2 r^4}{N^2(2n_b+n_f)}$ v.s. r at $g_{YM}^2 N/\pi = 1$. Left: (a) $\mathcal{N} = (1,1)$ SYM for $K = 4$ to 10 . Right: (b) $\mathcal{N} = (2,2)$ SYM for $K = 3, 4, 5, 6$.

$$= \left| \frac{L}{\pi} \langle 0 | T^{++}(k) | n \rangle \right|_{M_n=0}^2 \frac{6}{k^3 \pi^2 r^4}. \quad (25)$$

We have computed this quantity as a function of the inverse harmonic resolution $1/K$ and extrapolated to the continuum limit. The data currently available suggests that the non-zero contribution from these massless states persists in this limit.

2.1.2 The $\mathcal{N} = (2,2)$ theory. Let us now turn to the model with $\mathcal{N} = (2,2)$ supersymmetry. The SDLCQ version of this model was solved in Ref. [22]. The result of this computation can be inserted into Eq. (23). The result is shown in Fig. 3(b). This model appears to exhibit the onset of a gapless continuum of states more rapidly than the $\mathcal{N} = (1,1)$ model as the harmonic resolution K is increased. Just as we found in the latter model, this theory contains exactly massless states in the spectrum. These massless states appear to couple to $T^{++}|0\rangle$ only for even K , and the overlap appears to be decreasing for growing K . It is believed that this model is likely to exhibit a power law behavior c/r^γ for $\gamma > 4$ for the T^{++} correlator for $r \gg g_{YM}\sqrt{N}$ in the large N_c limit [4].

2.1.3 The $\mathcal{N} = (8,8)$ theory. In principle, we can now calculate the correlator numerically by evaluating Eq. (23). However, it turns out that even for very modest harmonic resolutions, we face a tremendous numerical task. At $K = 2, 3, 4$, the dimension of the associated Fock space is 256, 1632, and 29056, respectively. The usual procedure is to diagonalize the Hamiltonian P^- and then to evaluate the projection of each eigenfunction on the fundamental state $T^{++}(-K)|0\rangle$. Since we

are only interested in states which have nonzero value of such projection, we are able to significantly reduce our numerical efforts.

In the continuum limit, the result does not depend on which of the eight supercharges Q_α^- one chooses. In DLCQ, however, the situation is a bit subtler: while the spectrum of $(Q_\alpha^-)^2$ is the same for all α , the wave functions depend on the choice of supercharge [23]. This dependence is an artifact of the discretization and disappears in the continuum limit. What happens if we just pick one supercharge, say Q_1^- ? Since the state $T^{++}(-K)|0\rangle$ is a singlet under R-symmetry acting on the “flavor” index of Q_α^- , the correlator (23) does not depend on the choice of α even at finite resolution!

We can exploit this fact to simplify our calculations. Consider an operator S commuting with both P^- and $T^{++}(-K)$, and such that $S|0\rangle = s_0|0\rangle$. Then the Hamiltonian and S can be diagonalized simultaneously. We assume in the sequel that the set of states $|i\rangle$ is a result of such a diagonalization. In this case, only states satisfying the condition $S|i\rangle = s_0|i\rangle$ contribute to the sum in (23), and we only need to diagonalize P^- in this sector, which reduces the size of the problem immensely. We can deduce from the structure of the state $T^{++}(-K)|0\rangle$ that any transformation of the form

$$\begin{aligned} a_{ij}^I(k) &\rightarrow f(I)a_{ij}^{P[I]}(k), & f(I) &= \pm 1 \\ b_{ij}^\alpha(k) &\rightarrow g(\alpha)b_{ij}^{Q[\alpha]}(k), & g(\alpha) &= \pm 1 \end{aligned} \quad (26)$$

given arbitrary permutations P and Q of the 8 flavor indices, commutes with $T^{++}(-K)$. The vacuum will then be an eigenstate of this transformation with eigenvalue 1. The requirement for $P^- = (Q_1^-)^2$ to be invariant under S imposes some restrictions on the permutations. In fact, we will require that Q_1^- be invariant under S , in order to guarantee that P^- is invariant.

The form of the supercharge from [23] is

$$\begin{aligned} Q_\alpha^- &= \int_0^\infty [\dots] b_\alpha^\dagger(k_3) a_I(k_1) a_I(k_2) + \dots \\ &+ (\beta_I \beta_J^T - \beta_J \beta_I^T)_{\alpha\beta} [\dots] b_\beta^\dagger(k_3) a_I(k_1) a_J(k_2) + \dots \end{aligned} \quad (27)$$

Here the β_I are 8×8 real matrices satisfying $\{\beta_I, \beta_J^T\} = 2\delta_{IJ}$.

Let us consider the expression for Q_1^- , Eq. (27). The first part of the supercharge does not include β matrices, and is therefore invariant under the transformation, Eq. (26), as long as $g(1) = 1$ and $Q[1] = 1$. We will consider only such transformations. The crucial observation for the analysis of the symmetries of the β terms is that in the representation of the β matrices we have chosen, the expression $\mathcal{B}_{IJ}^\alpha = (\beta_I \beta_J^T - \beta_J \beta_I^T)_{1\alpha}$

may take only the values ± 2 or zero. Besides, for any pair (I, J) there is only one (or no) value of α corresponding to nonzero \mathcal{B} . Using this information, we may represent \mathcal{B} in a compact form. With the definition [24]

$$\mu_{IJ} = \begin{cases} \alpha, & \mathcal{B}_{IJ}^\alpha = 2 \\ -\alpha, & \mathcal{B}_{IJ}^\alpha = -2 \\ 0, & \mathcal{B}_{IJ}^\alpha = 0 \text{ for all } \alpha \end{cases}, \quad (28)$$

together with the special choice of β matrices we get the following expression for μ

$$\mu = \begin{pmatrix} 0 & 5 & -7 & 2 & -6 & 3 & -4 & 8 \\ -5 & 0 & -3 & 6 & 2 & -7 & 8 & 4 \\ 7 & 3 & 0 & -8 & -4 & -5 & 6 & 2 \\ -2 & -6 & 8 & 0 & -5 & 4 & 3 & 7 \\ 6 & -2 & 4 & 5 & 0 & -8 & -7 & 3 \\ -3 & 7 & 5 & -4 & 8 & 0 & -2 & 6 \\ 4 & -8 & -6 & -3 & 7 & 2 & 0 & 5 \\ -8 & -4 & -2 & -7 & -3 & -6 & -5 & 0 \end{pmatrix}.$$

The next step is to look for a subset of the transformations, Eq. (26), which satisfy the conditions $g(1) = 1$ and $Q[1] = 1$ and leave the matrix μ invariant. This invariance implies that

$$Q[\mu_{P[I]P[J]}] = g(\mu_{IJ})f(I)f(J)\mu_{IJ}. \quad (29)$$

The subset of transformations we are looking for forms a subgroup R of the permutation group $S_8 \times S_8$. Consequently, we will search for the elements of R that square to one. Products of such elements generate the whole group in the case of $S_8 \times S_8$. We will show later that this remains true for R . Not all of the Z_2 symmetries satisfying (29) are independent. In particular, if a and b are two such symmetries then aba is also a valid Z_2 symmetry. By going systematically through the different possibilities, we have found that there are 7 independent Z_2 symmetries in the group R . They are listed in Table 1. We explicitly constructed all the symmetries of the type, Eq. (26), which satisfy Eq. (29) using MATHEMATICA. It turns out that the group of such transformations has 168 elements, and we have shown that all of them can be generated from the seven Z_2 symmetries mentioned above.

In our numerical algorithm we implemented the Z_2 symmetries as follows. We can group the Fock states in classes and treat the whole class as a new state, because all states relevant for the correlator are singlets under the symmetry group R . As an example, consider the simplest non-trivial singlet

$$|1\rangle = \frac{1}{8} \sum_{I=1}^8 \text{tr} \left(a^\dagger(1, I) a^\dagger(K-1, I) \right) |0\rangle. \quad (30)$$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	b_2	b_3	b_4	b_5	b_6	b_7	b_8
1	a_7	a_3	a_2	a_6	a_8	a_4	a_1	a_5	b_2	$-b_3$	$-b_4$	$-b_6$	$-b_5$	b_8	b_7
2	a_3	a_6	a_1	a_5	a_4	a_2	a_8	a_7	$-b_4$	b_3	$-b_2$	$-b_5$	b_8	$-b_7$	b_6
3	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	$-b_3$	$-b_2$	b_4	$-b_5$	b_7	b_6	$-b_8$
4	a_5	a_4	a_8	a_2	a_1	a_7	a_6	a_3	$-b_2$	$-b_7$	b_8	b_5	$-b_6$	$-b_3$	b_4
5	a_8	a_3	a_2	a_7	a_6	a_5	a_4	a_1	$-b_5$	$-b_3$	b_7	$-b_2$	b_6	b_4	$-b_8$
6	a_5	a_8	a_7	a_6	a_1	a_4	a_3	a_2	$-b_8$	b_5	$-b_4$	b_3	$-b_6$	b_7	$-b_2$
7	a_4	a_6	a_8	a_1	a_7	a_2	a_5	a_3	$-b_2$	$-b_6$	b_5	b_4	$-b_3$	$-b_7$	b_8

Table 1. Seven independent Z_2 symmetries of the group R , which act on the 'flavor' quantum number of the different particles. Under the first of these symmetries, *e.g.*, the boson a_1 is transformed into a_7 , etc.

Hence, if we encounter the state $a^\dagger(1,1)a^\dagger(K-1,1)|0\rangle$ while constructing the basis, we will replace it by the class representative; in this case, by the state $|1\rangle$. Such a procedure significantly decreases the size of the basis, while keeping all the information necessary for calculating the correlator. In summary, this use of the discrete flavor symmetry of the problem reduces the size of the Fock space by orders of magnitude.

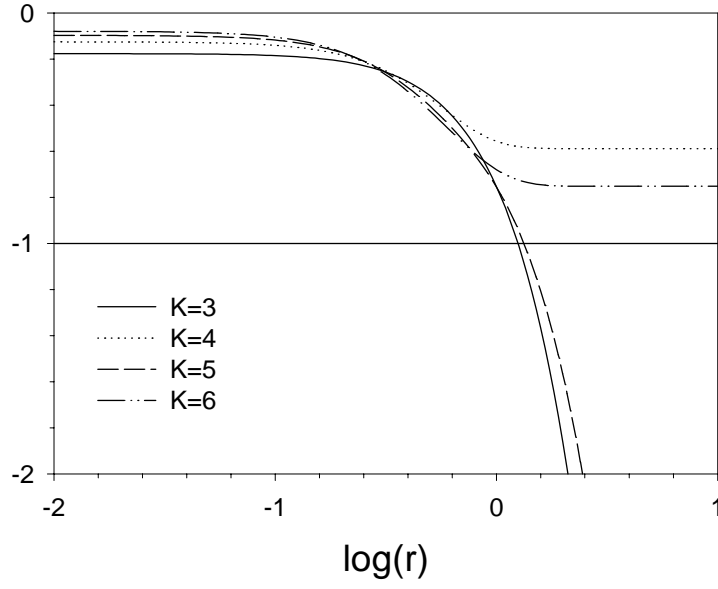
In addition to these simplifications, one can further improve on the numerical efficiency by using Lanczos diagonalization techniques[25]. Namely, we substitute the explicit diagonalization with an efficient approximation. The idea is to use a symmetry preserving (Lanczos) algorithm. If we start with a normalized vector $|u_1\rangle$ proportional to the fundamental state $T^{++}(-K)|0\rangle$, the Lanczos recursion will produce a tridiagonal representation of the Hamiltonian $H_{LC} = 2P^+P^-$. Due to orthogonality of $\{|u_i\rangle\}$, only the (1,1) element of the exponential of the tridiagonal matrix \hat{H}_{LC} will contribute to the correlator [26]. We exponentiate by diagonalizing $\hat{H}_{LC}\vec{v}_i = \lambda_i\vec{v}_i$ with eigenvalues λ_i and get

$$F(P^+, x^+) = \frac{|N_0|^{-2}}{2L} \left(\frac{\pi}{L}\right)^2 \sum_{j=1}^{N_L} |(v_j)_1|^2 e^{-i\frac{\lambda_j L}{2K\pi}x^+}.$$

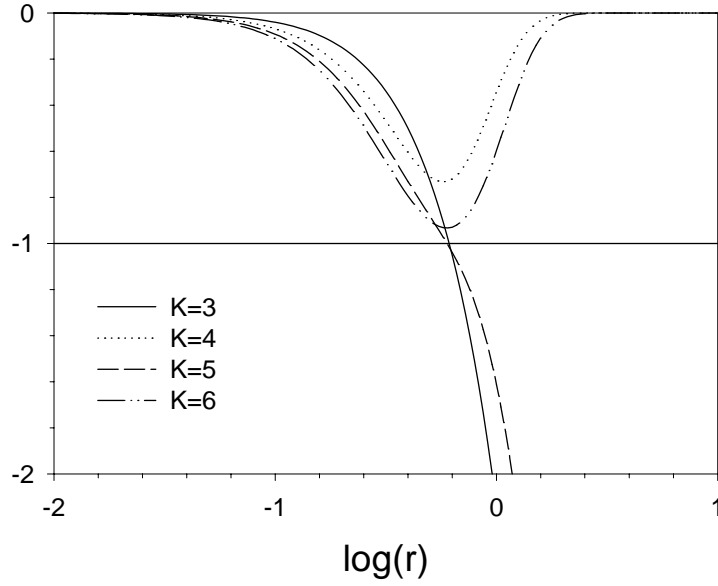
Finally, we Fourier transform to obtain

$$F(x^-, x^+) = \frac{1}{8\pi^2 K^3} \left(\frac{x^+}{x^-}\right)^2 \frac{1}{|N_0|^2} \sum_{j=1}^{N_L} |(v_j)_1|^2 \lambda_j^2 K_4(\sqrt{2x^+x^-}\lambda_j),$$

which is equivalent to Eq. (23). This algorithm is correct only if the number of Lanczos iterations N_L runs up to the rank of the original matrix, but *in praxi* already a basis of about 20 vectors covers all leading contributions to the correlator.



(a)



(b)

Figure 4. Top: (a) Log-Log plot of $\mathcal{F}(r) = \langle T^{++}(x)T^{++}(0) \rangle \left(\frac{x^-}{x^+} \right)^2 \frac{4\pi^2 r^4}{N_c^2(2n_b + n_f)}$ vs. r for $g_{YM}^2 N_c / \pi = 1.0$, $K = 3, 4, 5$ and 6 . Bottom: (b) the log-log derivative with respect to r of the correlation function in (a).

Numerical Results. To evaluate the expression for the correlator $\mathcal{F}(r)$, we have to calculate the mass spectrum and insert it into Eq. (23). In the $\mathcal{N} = (8, 8)$ supersymmetric Yang-Mills theory the contribution of massless states becomes a problem. These states exist in the SDLCQ calculation, but are unphysical. It has been shown that these states are not normalizable and that the number of partons in these states is even (odd) for K even (odd) [23]. Because the correlator is only sensitive to two particle contributions, the curves $\mathcal{F}(r)$ are different for even and odd K . Unfortunately, the unphysical states yield also the typical $1/r^4$ behavior, but have a wrong N_c dependence. The regular $1/r^4$ contribution is down by $1/N_c$, so we cannot see this contribution at large r , because we are working in the large N_c limit.

We can use this information about the unphysical states, however, to determine when our approximation breaks down. It is the region where the unphysical massless states dominate the correlator sum. Unfortunately, this is also the region where we expect the true large- r behavior to dominate the correlator, if only the extra states were absent. In Fig. 4(a) for even resolution, the region where the correlator starts to behave like $1/r^4$ at large r is clearly visible. In Fig. 4(b) we see that for even resolution the effect of the massless state on the derivative is felt at smaller values of r where the even resolution curves start to turn up. Another estimate of where this approximation breaks down, that gives consistent values, is the set of points where the even and odd resolution derivative curves cross. We do not expect these curves to cross on general grounds, based on work in [4], where we considered a number of other theories. Our calculation is consistent in the sense that this breakdown occurs at larger and larger r as K grows.

We expect to approach the line $d\mathcal{F}(r)/dr = -1$ line signaling the cross-over from the trivial $1/r^4$ behavior to the characteristic $1/r^5$ behavior of the supergravity correlator, Eq. (6). Indeed, the derivative curves in Fig. 4(b) are approaching -1 as we increase the resolution and appear to be about 85 – 90% of this value before the approximation breaks down. There is, however, no indication of convergence yet; therefore, we cannot claim a numerical proof of the Maldacena conjecture. A safe signature of equivalence of the field and string theories would be if the derivative curve would flatten out at -1 before the approximation breaks down.

2.2. Three-dimensional correlators

It remains a challenge to rigorously test the conjectured string/field theory correspondences. Although the so-called Maldacena conjecture

maybe the most exciting one, because it promises insight into full four-dimensional Yang-Mills theories in the strong coupling regime, there are other interesting scenarios. For instance, it was conjectured that the supergravity solutions corresponding to $p + 1$ SYM theories are black p -brane solutions, see *e.g.* Ref. [3]. Consequently, there are interesting testing scenarios also in three-dimensional spacetime. Numerically, of course, things get more difficult as the number of dimensions is increased. On the way to the full four-dimensional problem, it may be worthwhile to present our latest results on correlation functions in three dimensions; see also [13]. Fig. 7(b) shows the correlator for $\mathcal{N} = 1$ SYM(2+1) as a function of the distance r : it is converging well with the *transverse* cut-off T . To put things in perspective, we note that the construction of the largest Hamiltonian matrix involved in this calculation requires a Fock basis of approximately two million states. This is by a factor 100 more than we used in the test of the Maldacena conjecture described in this article, which itself was already substantially better than the first feasibility study [4].

The correlator of the energy momentum operator has been studied in conformal field theory in 2+1 dimensions [27], and this provides a reference point for our results. The structure of the correlators in conformal field theory is particularly simple in the collinear limit $x_\perp \rightarrow 0$, and we therefore find it convenient to work in this limit. From results in conformal field theory we expect that correlators behave as $1/r^6$ at small r , where we are probing deep inside the bound states. We have confirmed this $1/r^6$ behavior by an analytic calculation of the free-particle correlator in the DLCQ formalism [16].

The contributions of individual bound states have a characteristic length scale corresponding to the size of the bound states. On dimensional grounds one can show that the power behavior of the correlators are reduced by one power of r ; so for individual bound states the correlator behaves like $1/r^5$ for small r . It then becomes a nontrivial check to see that at small r the contributions of the bound states add up to give the expected $1/r^6$ behavior. We find this expected result as well as the characteristic rapid convergence of SDLCQ at both small and intermediate values of r .

At large r the correlator is controlled by the massless states of the theory. In this theory there are two types of massless states. At zero coupling all the states of the 1 + 1 dimensional theory are massless, and for non-vanishing coupling the massless states of the 1 + 1 theory are promoted to massless states of the 2 + 1 dimensional theory [17]. These states are BPS states and are exactly annihilated by one of the supercharges. This is perhaps the most interesting part of this calculation

because the BPS masses are protected by the exact supersymmetry of the numerical approximation and remain exactly zero at all couplings. Commonly in modern field theory one uses the BPS states to extrapolate from weak coupling to strong coupling. While the masses of BPS states remain constant as functions of the coupling, their wave functions certainly do not. The calculation of the correlator at large r provides a window to the coupling dependence of the BPS wave functions. We find, however, that there is a critical coupling where the correlator goes to zero, which depends on the transverse resolution. A detailed study of this critical coupling shows that it goes to infinity linearly with the square root of the transverse resolution. Below the critical coupling the correlator converges rapidly at large r . One possible explanation is that this singular behavior signals the breakdown of the SDLCQ calculation for the BPS wave function at couplings larger than the critical coupling. If this is correct, calculation of the BPS wave function at stronger couplings requires higher transverse resolutions. We note that above the critical coupling (see Fig. 7 below) we do find convergence of the correlator at large r but at a significantly slower rate.

Let us now return to the details of the calculation. We would like to compute a general expression of the form

$$F(x^+, x^-, x^\perp) = \langle 0 | T^{++}(x^+, x^-, x^\perp) T^{++}(0, 0, 0) | 0 \rangle. \quad (31)$$

Here we will calculate the correlator in the collinear limit, that is, where $x^\perp = 0$. We know from conformal field theory [27] calculations that this will produce a much simpler structure.

The calculation is done by inserting a complete set of intermediate states $|\alpha\rangle$,

$$F(x^+, x^-, x^\perp = 0) = \sum_{\alpha} \langle 0 | T^{++}(x^-, 0, x^\perp = 0) | \alpha \rangle e^{-iP_{\alpha}^- x^+} \langle \alpha | T^{++}(0, 0, 0) | 0 \rangle. \quad (32)$$

with energy eigenvalues P_{α}^- . The momentum operator $T^{++}(x)$ is given by

$$T^{++}(x) = \text{tr} \left[(\partial_- \phi)^2 + \frac{1}{2} (i\psi \partial_- \psi - i(\partial_- \psi) \psi) \right] = T_B^{++}(x) + T_F^{++}(x). \quad (33)$$

In terms of the mode operators, we find

$$T^{++}(x^+, x^-, 0) | 0 \rangle = \frac{1}{2Ll} \sum_{n,m} \sum_{n_\perp, m_\perp} T(n, m) e^{-i(P_n^+ + P_m^+) x^-} | 0 \rangle, \quad (34)$$

where the boson and fermion contributions are given by

$$\frac{L}{\pi} T_B^{++}(n, m) |0\rangle = \frac{\sqrt{nm}}{2} \text{tr} \left[a_{ij}^\dagger(n, n_\perp) a_{ji}^\dagger(m, m_\perp) \right] |0\rangle \quad (35)$$

and

$$\frac{L}{\pi} T_F^{++}(n, m) |0\rangle = \frac{(n-m)}{4} \text{tr} \left[b_{ij}^\dagger(n, n_\perp) b_{ji}^\dagger(m, m_\perp) \right] |0\rangle. \quad (36)$$

Given each $|\alpha\rangle$, the matrix elements in (32) can then be evaluated, and the sum computed.

First, however, it is instructive to do the calculation where the states $|\alpha\rangle$ are a set of free particles with mass m . The boson contribution is

$$F(x^+, x^-, 0)_B = \sum_{n, m, s, t} \left(\frac{\pi}{4L^2 l} \right)^2 e^{-iP_n^- x^+ - iP_n^+ x^- - iP_m^- x^+ - iP_m^+ x^-} \times \sqrt{mnst} \langle 0 | \text{tr} [a(n, n_\perp) a(m, m_\perp)] \text{tr} [a^\dagger(s, s_\perp) a^\dagger(t, t_\perp)] | 0 \rangle, \quad (37)$$

where the sum over n implies sums over both n and n_\perp , and

$$P_n^- = \frac{m^2 + (2n_\perp \pi / l)^2}{2n\pi / L} \quad \text{and} \quad P_n^+ = \frac{n\pi}{L}. \quad (38)$$

The sums can be converted to integrals which can be explicitly evaluated, and we find

$$F(x^+, x^-, 0)_B = \frac{i}{2(2\pi)^3} m^5 \left(\frac{x^+}{x^-} \right)^2 \frac{1}{x} K_{5/2}^2(mx), \quad (39)$$

where $x^2 = 2x^+ x^-$. Similarly for the fermions we find

$$F(x^+, x^-, 0)_F = \sum_{n, m, s, t} \left(\frac{\pi}{8L^2 l} \right)^2 e^{-iP_n^- x^+ - iP_n^+ x^- - iP_m^- x^+ - iP_m^+ x^-} \times (m-n)(s-t) \langle 0 | \text{tr} [b(n, n_\perp) b(m, m_\perp)] \text{tr} [b^\dagger(s, s_\perp) b^\dagger(t, t_\perp)] | 0 \rangle. \quad (40)$$

After doing the integrals we obtain

$$F(x^+, x^-, 0)_F = \frac{i}{4(2\pi)^3} \frac{m^5}{x} \left(\frac{x^+}{x^-} \right)^2 \left[K_{7/2}(mx) K_{3/2}(mx) - K_{5/2}^2(mx) \right]. \quad (41)$$

We can continue to Euclidean space by taking $r = \sqrt{2x^+ x^-}$ to be real, and, finally, in the small- r limit we find

$$\left(\frac{x^-}{x^+} \right)^2 F(x^+, x^-, 0) = \frac{-3i}{8(2\pi)^2} \frac{1}{r^6}, \quad (42)$$

which exhibits the expected $1/r^6$ behavior.

Now let us return to the calculation using the bound-state solution obtained from SDLCQ. It is convenient to write

$$F(x^+, x^-, 0) = \sum_{n,m,s,t} \left(\frac{\pi}{2L^2 l} \right)^2 \langle 0 | \frac{L}{\pi} T(n, m) e^{-iP_{op}^- x^+ - iP^+ x^-} \frac{L}{\pi} T(s, t) | 0 \rangle, \quad (43)$$

where P_{op}^- is the Hamiltonian operator. We again insert a complete set of bound states $|\alpha\rangle$ with light-cone energies $P_\alpha^- = (M_\alpha^2 + P_\perp^2)/P^+$ at resolution K (and therefore $P^+ = \pi K/L$) and with total transverse momentum $P_\perp = 2N_\perp \pi/l$. We also define

$$|u\rangle = N_u \frac{L}{\pi} \sum_{n,m} \delta_{n+m,K} \delta_{n_\perp+m_\perp, N_\perp} T(n, m) |0\rangle, \quad (44)$$

where N_u is a normalization factor such that $\langle u|u\rangle = 1$. It is straightforward to calculate the normalization, and we find

$$\frac{1}{N_u^2} = \frac{K^3}{8} \left(1 - \frac{1}{K}\right) (2T + 1). \quad (45)$$

The correlator (43) becomes

$$F(x^+, x^-, 0) = \sum_{K, N_\perp, \alpha} \left(\frac{\pi}{2L^2 l} \right)^2 e^{-iP_\alpha^- x^+ - iP^+ x^-} \frac{1}{N_u^2} |\langle u|\alpha\rangle|^2. \quad (46)$$

We will calculate the matrix element $\langle u|\alpha\rangle$ at fixed longitudinal resolution K and transverse momentum $N_\perp = 0$. Because of transverse boost invariance the matrix element does not contain any explicit dependence on N_\perp . To leading order in $1/K$ the explicit dependence of the matrix element on K is K^3 ; it also contains a factor of l , the transverse length scale. To separate these dependencies, we write F as

$$F(x^+, x^-, 0) = \frac{1}{2\pi} \sum_{K, N_\perp, \alpha} \frac{1}{2L} \frac{1}{l} \left(\frac{\pi K}{L} \right)^3 e^{-iP_\alpha^- x^+ - iP^+ x^-} \frac{|\langle u|\alpha\rangle|^2}{l K^3 |N_u|^2}. \quad (47)$$

We can now do the sums over K and N_\perp as integrals over the longitudinal and transverse momentum components $P^+ = \pi K/L$ and $P^\perp = 2\pi N_\perp/l$. We obtain

$$\frac{1}{\sqrt{-i}} \left(\frac{x^-}{x^+} \right)^2 F(x^+, x^-, 0) = \sum_\alpha \frac{1}{2(2\pi)^{5/2}} \frac{M_\alpha^{9/2}}{\sqrt{r}} K_{9/2}(M_\alpha r) \frac{|\langle u|\alpha\rangle|^2}{l K^3 |N_u|^2}. \quad (48)$$

In practice, the full sum over α is approximated by a Lanczos [25] iteration technique [24, 26] that eliminates the need for full diagonalization of the Hamiltonian matrix. For the present case, the number of iterations required was on the order of 1000.

Looking back at the calculation for the free particle, we see that there are two independent sums over transverse momentum, after the contractions are performed. One would expect that the transverse dimension is controlled by the dimensional scale of the bound state (R_B) and therefore the correlation should scale like $1/r^4 R_B^2$. However, because of transverse boost invariance, the matrix element must be independent of the difference of the transverse momenta and therefore must scale as $1/r^5 R_B$.

There are three commuting Z_2 symmetries. One of them is the parity in the transverse direction,

$$P : a_{ij}(k, n^\perp) \rightarrow -a_{ij}(k, -n^\perp), \quad b_{ij}(k, n^\perp) \rightarrow b_{ij}(k, -n^\perp). \quad (49)$$

The second symmetry [28] is with respect to the operation

$$S : a_{ij}(k, n^\perp) \rightarrow -a_{ji}(k, n^\perp), \quad b_{ij}(k, n^\perp) \rightarrow -b_{ji}(k, n^\perp). \quad (50)$$

Since P and S commute with each other, we need only one additional symmetry $R = PS$ to close the group. Since Q^- , P and S commute with each other, we can diagonalize them simultaneously. This allows us to diagonalize the supercharge separately in the sectors with fixed P and S parities and thus reduce the size of matrices. Doing this one finds that the roles of P and S are different. While all the eigenvalues are usually broken into non-overlapping S -odd and S -even sectors [29], the P symmetry leads to a double degeneracy of massive states (in addition to the usual boson-fermion degeneracy due to supersymmetry).

Numerical Results. The first important numerical test is the small- r behavior of the correlator. Physically we expect that at small r the bound states should behave as free particles, and therefore the correlator should have the behavior of the free particle correlator which goes like $1/r^6$. We see in (48) that the contributions of each of the bound states behaves like $1/r^5$. Therefore, to get the $1/r^6$ behavior of the free theory, the bound states must work in concert at small r . It is clear that this cannot work all the way down to $r = 0$ in the numerical calculation. At very small r the most massive state allowed by the numerical approximation will dominate, and the correlator must behave like $1/r^5$. To see what happens at slightly larger r it is useful to consider the behavior at small coupling. There, the larger masses go like

$$M_\alpha \simeq \sum_i \frac{(k_i^\perp)^2}{2P^+}. \quad (51)$$

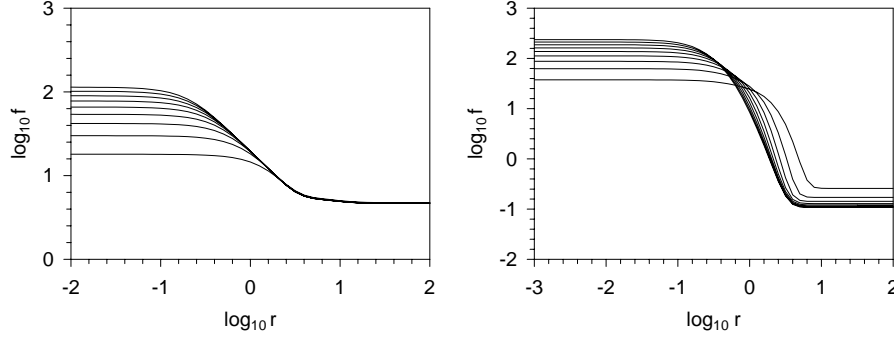


Figure 5. The log-log plot of the correlation function $f \equiv r^5 \langle T^{++}(x) T^{++}(0) \rangle \left(\frac{x^-}{x^+} \right)^2 \frac{16\pi^3 K^3 l}{105 \sqrt{-i}}$ vs. r . Left: (a) in units where $g = g_{YM} \sqrt{N_c l} / 2\pi^{3/2} = 0.10$ for $K = 4$ and $T = 1$ to 9 ; Right: (b) in units where $g = g_{YM} \sqrt{N_c l} / 2\pi^{3/2} = 1$ for $K = 5$ and $T = 1$ to 9 .

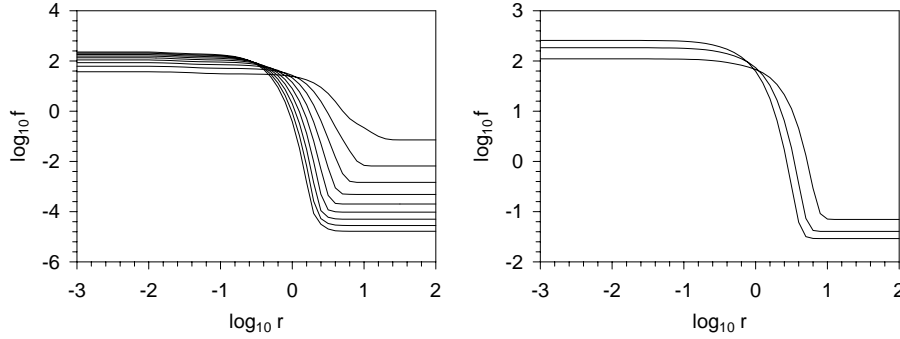


Figure 6. Left: (a) Same as Fig. 5(b), but for $g = g_{YM} \sqrt{N_c l} / 2\pi^{3/2} = 10$. Right: (b) Log-Log plot of the *three-dimensional* correlation function $f \equiv r^5 \langle T^{++}(x) T^{++}(0) \rangle \left(\frac{x^-}{x^+} \right)^2 \frac{16\pi^3 K^3 l}{105 \sqrt{-i}}$ vs. r for $g = g_{YM}^2 \sqrt{N_c l} / 2\pi^{3/2} = 1.0$ for $K = 6$ and $T = 1$ to 5 .

Consequently, as we remove the k^\perp cutoff, *i.e.* increase the transverse resolution T , more and more massive bound states will contribute, and the dominant one will take over at smaller and smaller r leading to the expected $1/r^6$. This is exactly what we see happening in Fig. 5 at weak coupling with longitudinal resolution $K = 4$ and 5 .

The correlator converges from below at small r with increasing T , and in the region $-0.5 \leq \log r \leq 0.5$ the plot of r^5 times the correlator falls like $1/r$. In Fig. 7 at resolution $K = 5$ we see the same behavior for strong coupling ($g = g_{YM} \sqrt{N_c l} / 2\pi^{3/2} = 10$) but now at smaller r ($\log r \simeq -0.5$) as one would expect. Again at strong coupling we see that

the correlator converges quickly and from below in T . All indications are that at small r the correlators are well approximated by SDLCQ, converge rapidly, and show the behavior that one would expect on general physical grounds. This gives us confidence to go on to investigate the behavior at large r .

The behavior for large r is governed by the massless states. From earlier work [30, 17] on the spectrum of this theory we know that there are two types of massless states. At $g = 0$ the massless states are a reflection of all the states of the dimensionally reduced theory in $1 + 1$. In $2+1$ dimensions these states behave as $g^2 M_{1+1}^2$. We expect therefore that for $g \simeq 0$ there should be no dependence of the correlator on the transverse momentum cutoff T at large r . In Fig. 5(a) this behavior is clearly evident.

At all couplings there are exactly massless states which are the BPS states of this theory, which has zero central charge. These states are destroyed by one supercharge, Q^- , and not the other, Q^+ . From earlier work [30] on the spectrum we saw that the number of BPS states is independent of the transverse resolution and equal to $2K - 1$. Since these states are exactly massless at all resolutions, transverse and longitudinal convergence of these states cannot be investigated using the spectrum. These states do have a complicated dependence on the coupling g through their wave function, however. This is a feature so far not encountered in DLCQ [16]. In previous DLCQ calculations one always looked to the convergence of the spectrum as a measure of the convergence of the numerical calculation. Here we see that it is the correlator at large r that provides a window to study the convergence of the wave functions of the BPS states. In Fig. 7 we see that the correlator converges from above at large r as we increase T .

We also note that the correlator at large r is significantly smaller than at small r , particularly at strong coupling. In our initial study of the BPS states [17] we found that at strong coupling the average number of particles in these BPS states is large. Therefore the two particle components, which are the only components the T^{++} correlator sees, are small.

The coupling dependence of the large- r limit of the correlator is much more interesting than we would have expected based on our previous work on the spectrum. To see this behavior we study the large- r behavior of the correlator at fixed g as a function of the transverse resolution T and at fixed T as a function of the coupling g . We see a hint that something unusual is occurring in Fig. 7. For values of the coupling up to about $g = 1$ we see the typical rapid convergence in the transverse momentum cutoff; however, at larger coupling the convergence appears

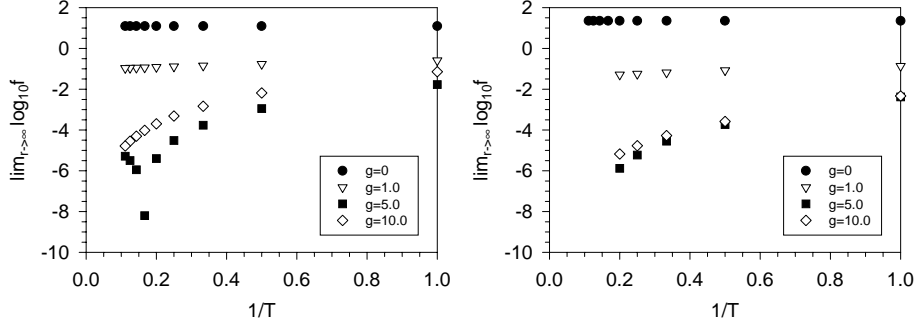


Figure 7. The large- r limit of the log of the correlation function $f \equiv r^5 \langle T^{++}(x) T^{++}(0) \rangle \left(\frac{x^-}{x^+} \right)^2 \frac{16\pi^3}{105} \frac{K^3 l}{\sqrt{-i}}$ vs. $1/T$ for [left](a) $K = 5$ and [right](b) $K = 6$ and for various values of the coupling $g = g_{\text{YM}} \sqrt{N_c l} / 2\pi^{3/2}$.

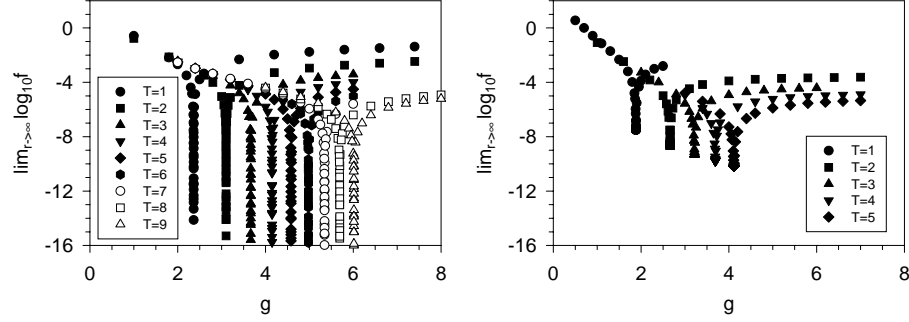


Figure 8. The large- r limit of the correlation function $f \equiv r^5 \langle T^{++}(x) T^{++}(0) \rangle \left(\frac{x^-}{x^+} \right)^2 \frac{16\pi^3}{105} \frac{K^3 l}{\sqrt{-i}}$ vs. $g = g_{\text{YM}} \sqrt{N_c l} / 2\pi^{3/2}$ for [left](a) $K = 5$ [right](b) $K = 6$ and for various values of the transverse resolution T .

to deteriorate, and we see that for $g = 5$ the correlator is smaller than at $g = 10$. We see this same behavior at both $K = 5$ and $K = 6$. We do not see this behavior at $K = 4$, but it is not unusual for effects to appear only at a large enough resolution in SDLCQ. In Fig. 8 we see that the correlator does not in fact decrease monotonically with g but rather has a singularity at a particular value of the coupling which is a function of K and T . Beyond the singularity the correlator again appears to behave well.

If we plot the ‘critical’ couplings, at which the correlator goes to zero, versus \sqrt{T} , as in Fig. 9, we see that they lie on a straight line, *i.e.* this coupling is a linear function of \sqrt{T} in both cases, $K = 5$ and

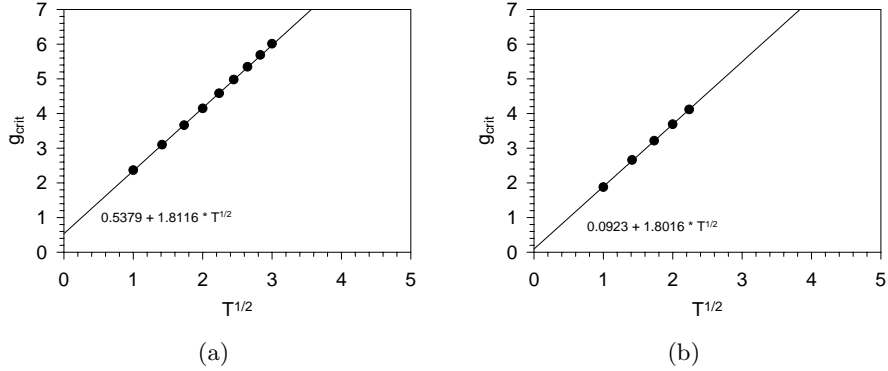


Figure 9. Critical coupling g_{crit} versus \sqrt{T} for (a) $K = 5$ and (b) $K = 6$.

6. Consequently, the ‘critical’ coupling goes to infinity in the transverse continuum limit. It appears as though we have encountered a finite transverse cutoff effect. The most likely conclusion is that our numerical calculation of the BPS wave function is only valid for $g < g_{\text{crit}}(T)$. While the large- r correlator does converge above the critical coupling, it is unclear at this time if it has any significance. It might have been expected that one would need larger and larger transverse resolution to probe the strong coupling region, the occurrence of the singular behavior that we see is a surprise, and we have no detailed explanation for it at this time. We see no evidence of a singular behavior at small or intermediate r . This indicates, but does not prove, that our calculations of the massive bound states is valid at all g .

We do not seem to see a region dominated by the massive bound states, that is, a region where r is large enough that we see the structure of the bound states but small enough that the correlator is not dominated by the massless states of the theory. Such a region might give us other important information about this theory.

3. Conclusions

In this note we reported on progress in an attempt to rigorously test the conjectured equivalence of two-dimensional $\mathcal{N} = (8, 8)$ supersymmetric Yang-Mills theory and a system of $D1$ -branes in string theory. Within a well-defined non-perturbative calculation, we obtained results that are within 10-15% of results expected from the Maldacena conjecture. The results are still not conclusive, but they definitely point in right direction. Compared to previous work [4], we included orders of magnitude more states in our calculation and thus greatly improved the

testing conditions. We remark that improvements of the code and the numerical method are possible and under way. During the calculation we noticed that contributions to the correlator come from only a small number of terms. An analytic understanding of this phenomenon would greatly accelerate calculations. We point out that in principle we could study the proper $1/r$ behavior at large r by computing $1/N_c$ corrections, but this interesting calculation would mean a huge numerical effort.

In this work we also discussed the calculation of the stress-tensor correlator $\langle 0|T^{++}(x)T^{++}(0)|0\rangle$ in $\mathcal{N} = 1$ SYM in 2+1 dimension at large N_c in the collinear limit. We find that the free-particle correlator behaves like $1/r^6$, in agreement with results from conformal field theory. The contribution from an individual bound state is found to behave like $1/r^5$, and at small r such contributions conspire to reproduce the conformal field theory result $1/r^6$. We do not seem to find an intermediate region in r where the correlator behaves as $1/r^5$, reflecting the behavior of the individual massive bound states.

At large r the correlator is dominated by the massless BPS states of the theory. We find that as a function of g the large- r correlator has a critical value of g where it abruptly drops to zero. We have investigated this singular behavior and find that at fixed longitudinal resolution the critical coupling grows linearly with \sqrt{T} . We conjecture that this critical coupling signals the breakdown of SDLCQ at sufficiently strong coupling at fixed transverse resolution, T . While this might not be surprising in general, it is surprising that the behavior appears in the BPS wave functions and that we see no sign of this behavior in the massive states. We find that above the critical coupling the correlator still converges but significantly slower. It is unclear at this time if we should attach any significance to the correlator in this region.

This calculation emphasizes the importance of BPS wave functions which carry important coupling dependence, even though the mass eigenvalues are independent of the coupling. We will discuss the spectrum, the wavefunctions and associated properties of all the low energy bound states of $\mathcal{N} = 1$ SYM in 2+1 dimensions in a subsequent paper [31].

A number of computational improvements have been implemented in our code to allow us to make these detailed calculations. The code now fully utilizes the three known discrete symmetries of the theory, namely supersymmetry, transverse parity P , Eq. (49), and the Z_2 symmetry S , Eq. (50). This reduces the dimension of the Hamiltonian matrix by a factor of 8. Other, more efficient storage techniques allow us to handle on the order of 2,000,000 states in this calculation, which has been performed on a single processor Linux workstation. Our improved storage techniques should allow us to expand this calculation to include

higher supersymmetries without a significant expansion of the code or computational power. We remain hopeful that porting to a parallel machine will allow us to tackle problems in full 3+1 dimensions.

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